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Financial laws with algebraic automata theory

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Abstract

This paper shows that the concept of financial law has the structure of automaton. It is then shown that the financial law impose a group structure to the monoid of automaton and there are obtained, in natural way, the concepts of stationary, stationary of order n and dynamic financial laws, proving two algebraic characterization of the first. Finally, it is introduced the concept of $\overline{a}$-stationary financial law and its applications.

Key-words: Financial law; Semiautomata; Automata; Monoid; Group; Stationary; Dynamic.

Resumen

En este artículo se demuestra que el concepto de ley financiera tiene la estructura de autómata. Se prueba entonces que la ley financiera impone la estructura de grupo al monoide del autómata, obteniéndose, de manera natural, los conceptos de ley financiera estacionaria, estacionaria de orden n y dinámica, demostrándose dos caracterizaciones algebráicas de las primeras. Finalmente, se introduce el concepto de ley financiera $\overline{a}$-estacionaria, así como sus aplicaciones.

Palabras-clave: Ley financiera; Semiautómata; Autómata; Monoide; Grupo; Estacionario; Dinámico.

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1 Introduction

It is shown that a certain similarity exists between economic activity and formal computation (Ames (1983)). This parallelism can be seen in the Mathematics of Finance, whose predecessors are the Italian writers Cantelli, Insolera, Bonferroni, Levi, etc..

The semi-axiomatic stage in the Mathematics of Finance starts with Insolera and there is no doubt that its subsequent influence has been noticed. In his work "Corso di Matematica Finanziaria" (Insolera (1937)), in which he quotes Fisher, he states a difference between the *state* of an element of wealth, which is capital and the *movement* of an element of wealth during a period of time, which is *interest*.

According to Fisher (1912), interest is an *abstract device*, a stream, an *action* that capital endures in time, producing modifications to itself. In this way Insolera defines capital as "all the money invested in a financial operation, understanding that all such action determines a quantitative variation of capital".

On the other hand, in reality there seems to be some coincidence in that the point of departure in the concept of Financial Law has to be the theory of relations of preference, which is based on two fundamental concepts (Bhaumik and Das (1984)):

1. *Profits*: A profit is the satisfaction produced by a good or a service characterized by its physical properties, its location and by the date from when it is available.

   The measurable and quantifiable character of profit han been criticized and discussed by various authors like Hicks and Allen, who elaborated their theory without considering that profit was measurable, on the basis that only the consumer is capable of establishing a scale of preferences, thus giving rise to a so-called ordinal approach.

2. *Preference relations*: An economic agent prefers the vector $\overrightarrow{x}$ to the vector $\overrightarrow{y}$, if it is necessary for him to choose $\overrightarrow{x}$, provided that he is offered the alternatives $\overrightarrow{x}$ and $\overrightarrow{y}$.

The previous relations can fulfil a series of axioms (Bhaumik and Das (1984)) of which are chosen, in order to lay the foundations of the Mathematics of Finance, those that best represent the reality of financial operations. Basically the idea consist of considering two goods:

1. The *quantity* (C) or expression in monetary units of a product or service.

2. The *time* (t) or moment of availability or expiration of a said product or service.

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It is logical to think that the quantity C_0 considered in the actual moment t_0, in order that it remains "equivalent" within one year, has to increase its value giving rise to another quantity $C > C_0$. Therefore, it doesn't happen here like in the general theory in which there is a "diversion" from one product to another, but completely the opposite, that's why we will say that time is a negative economic good (M. A. Gil and L. Gil (1987)) and this will cause what is called the principle of sub-estimation of future capital (Gil (1987)), giving rise to increasing curves of indifference in place of decreasing ones.

It is also logical to exact that the relation of preference ($\preceq$) fulfils the reflexive, transitive and complete or decisive properties, that provides us with the relation of induced indifference ($\sim$) which fulfils the reflexive, symmetrical and transitive properties, that is to say it is an equivalence relation (Rodríguez (1984)).

Thus we come to the conclusion that the classes of equivalence caused by the relation $\sim$ are but curves of financial indifference with which the quotient set will be the "map of indifference". This means that, given a curve of financial indifference, "we can move freely along it" (the same as a free vector along the plane) obtaining in each instant equivalent financial capitals between them.

From this construction we can generate financial laws from a map of financial indifference and the opposite should also occur.

In this approach to the concept of financial law let us consider, in principle, time as a discreet variable, with which a real map of indifference will not exist but it will exist with a group of formed curves at isolated points which will be more or less separated depending on the size of the period.

This should not be a disadvantage since, in case of being interested in calculating an intermediate equivalent capital, we can employ another financial law with an adequate period, belonging to the same or distinct family as the initial law, with the one which, in this way we will be able to enter whatever point on the temporary axis not covered by the route of the initial period.

However, in order to formalize the previous idea inside our approach, we need a mathematical instrument different to those used up until now (basically the concept of function$^{(1)}$) and this is favoured by the following motives:

1. The idea of period (time), as a difference between the expiry dates of the final capital and the initial capital, has a different nature that the true expiry, because this is a reference whose value can or cannot (depending on dynamic and stationary financial systems respectively$^{(2)}$) intervene in the expression of the financial system. However, the period is a size that acts on expirations, increasing them or decreasing them, for which some writers (Gil (1987)) also call it "internal time". This difference between the time as period and as expiry doesn't appear clearly in the classical models of financial laws.

2. The concept of financial law is based in a set of properties, one of them is the homogeneity of first-degree concerning the quantity. However, financial practice is introducing operations, such as "highly remunerated current accounts"(3), in which that homogeneity is casted doubt (De Pablo (1993)). In fact, the projection or substitute in an instant p from a "sufficiently high quantity" doesn't coincide with the product of that quantity by the projection of the monetary unit. After saying this, it could be thought that a "super-account" suposses the application of some independent financial laws set, depending on the quantity in each case. However, this interpretation would suposse a "restricted" financial law to an interval of quantities, which would obstruct the construction of financial processes (Gil (1987)), having as base that financial law, because, depending on the quantity, that process could include more or less factors.

   On the other hand, in the financial operations described before, by general, an interval without remunerating or exemption exists, which gives the posibility that financial laws are defined on an strict subset of the positive real numbers set.

   The first of the facts described origines that we can not calculate the equivalent of a financial capital by consecutive multiplications of C by $F(1,t;p)$ but that we have to compose the quantity C in $F(C,t;p)$ consecutively.

3. Mathematically, this procedure presents some difficulties, among other reasons by the fact that $F(C,t;p)$ is a function of $\mathcal{R}^3$ into $\mathcal{R}$ and, to make this composition, we would come to expressions as follows:

$$F[F(C,t,t'),t'),t''],$$

   which is not easily manageable.

   However these difficulties can be solvented using the *Algebraic Theory of Automata* (see Arbib (1964), Arbib (1968a), Arbib (1968b), Cohn (1975), Booth (1967), Eilenberg (1974), Ginzburg (1968), Holcombe (1987), Lidl and Pilz (1984)), as through this instrument, on the one hand, we can separate quantities and, on the other hand, we can do the same with the expirations and, separately, make an study from which, in a natural way, properties and generalizated caracterizations appear from the simply and amply additive and multiplicative systems(4); besides the stationary.

   Therefore, we need to restrict the expirations to a discreet set, what, moreover, doesn't represent any generality lost, because it is the way in which it is normally worked (Cruz (1994)).

This paper is organized as follows: the next section introduces the model applied, presenting the concepts of semiautomaton, automaton and series composition. Moreover, in the third section, axioms of preference in a rational choice are presented, which justifies the definition of financial law in the fourth section. The fifth section includes the algebraic properties of financial law and finally a classification is reported with two theorems of characterization of stationary financial laws.

2 Semiautomata and automata

2.1 Definition (semiautomaton)

A *semiautomaton* is a triple

$$\mathcal{S} = (Z, A, \delta)$$

consisting of two nonempty sets Z and A and a function(5)

$$\delta : Z \times A \longrightarrow Z.$$

Z is called the *set of states*, A the *input alphabet* and δ the *next-state function* of $\mathcal{S}$.

2.2 Definition (automaton)

A *automaton* is a quintupel

$$\mathcal{A} = (Z, A, B, \delta, \lambda)$$

where

$$(Z, A, \delta)$$

is a semiautomaton, B is a nonempty set called *output alphabet* and

$$\lambda : Z \times A \longrightarrow B$$

is the *output function*.

If $z \in Z$ and $a \in A$, then we interpret $\delta(z, a) \in Z$ as the next state into which z is transformed by the input a. $\lambda(z, a) \in B$ is the output of z resulting from the input a. Thus if the automaton is in state z and receives input a, then it changes to state $\delta(z, a)$ with output $\lambda(z, a)$.

Let us consider the set $\overline{A} = F_A$ of the words (the empty word included) that we can write with the elements (letters) of a set A. We define in $\overline{A}$ an operation $\top$ in the following way. Let be $\overline{a} = a_1 a_2 .....a_p$, $\overline{b} = b_1 b_2 .....b_q$:

$$\overline{a} \top \overline{b} = a_1 a_2 .....a_p b_1 b_2 .....b_q .$$

Obviously, $\top$ is an internal operation in $\overline{A}$, is associative and the empty word Λ is the identity element, because:

$$\overline{a} \top \Lambda = \Lambda \top \overline{a} = \overline{a}, \ \forall \overline{a} \in \overline{A}.$$

The monoid $\overline{A}$ is called the *free monoid on* A.

In our study of automata we extend the input set A to the free monoid $\overline{A} = F_A$, with Λ as identity.We also extend δ and λ from $Z \times A$ to $Z \times \overline{A}$ by defining for $z \in Z$ and $a_1, a_2, \ldots.., a_r \in A$:

$$\overline{\delta}(z, \Lambda) = z,$$

$$\overline{\delta}(z, a_1) = \delta(z, a_1),$$

$$\overline{\delta}(z, a_1 a_2) = \delta(\overline{\delta}(z, a_1), a_2),$$

$$\vdots$$

$$\overline{\delta}(z, a_1 a_2 \ldots.. a_r) = \delta(\overline{\delta}(z, a_1 a_2 \ldots.. a_{r-1}), a_r),$$

and

$$\overline{\lambda}(z, \Lambda) = \Lambda,$$

$$\overline{\lambda}(z, a_1) = \lambda(z, a_1),$$

$$\overline{\lambda}(z, a_1 a_2) = \lambda(z, a_1)\overline{\lambda}(\delta(z, a_1), a_2),$$

$$\vdots$$

$$\overline{\lambda}(z, a_1 a_2 \ldots.. a_r) = \lambda(z, a_1)\overline{\lambda}(\delta(z, a_1), a_2 a_3 \ldots.. a_r).$$

In this way we obtain functions $\overline{\delta} : Z \times \overline{A} \longrightarrow Z$ and $\overline{\lambda} : Z \times \overline{A} \longrightarrow \overline{B}$. The semiautomaton $\mathcal{S} = (Z, A, \delta)$ (respectively, the automaton $\mathcal{A} = (Z, A, B, \delta, \lambda)$) is thus generalized to the new semiautomaton $\overline{\mathcal{S}} = (Z, \overline{A}, \overline{\delta})$ (respectively, automaton $\overline{\mathcal{A}} = (Z, \overline{A}, \overline{B}, \overline{\delta}, \overline{\lambda})$).

2.3 Definition (subautomaton)

$\mathcal{A}_1 = (Z_1, A, B, \delta_1, \lambda_1)$ is called a *subautomaton* of $\mathcal{A}_2 = (Z_2, A, B, \delta_2, \lambda_2)$ ($\mathcal{A}_1 \leq \mathcal{A}_2$) if $Z_1 \subseteq Z_2$ and δ_1 and λ_1 are the restrictions of δ_2 and λ_2, respectively, on $Z_1 \times A$.

Analogously, *subsemiautomaton* is defined.

Let $\mathcal{S} = (Z, A, \delta)$ be a semiautomaton. We consider $\overline{\mathcal{S}} = (Z, \overline{A}, \overline{\delta})$.

2.4 Notation

$$\forall \overline{a} \in \overline{A}, \text{ let } f_{\overline{a}} : Z \longrightarrow Z /\ z \mapsto f_{\overline{a}}(z) = \overline{\delta}(z, \overline{a}).$$

2.5 Theorem

$M_{\mathcal{S}} = (\{f_{\overline{a}} /\ \overline{a} \in \overline{A}\}, \circ)$ is a monoid.

Proof. See R. Lidl and G. Pilz (1984).

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2.6 Equivalence relation on $\overline{A}$

For $\overline{a} \in \overline{A}$, let $f_{\overline{a}} : Z \longrightarrow Z,\ z \longmapsto \overline{\delta}(z, \overline{a})$.

$$\overline{a}_1 \equiv \overline{a}_2 \Leftrightarrow f_{\overline{a}_1} = f_{\overline{a}_2} \Leftrightarrow$$

$$\Leftrightarrow \forall z \in Z,\ f_{\overline{a}_1}(z) = f_{\overline{a}_2}(z) \Leftrightarrow \forall z \in Z,\ \overline{\delta}(z, \overline{a}_1) = \overline{\delta}(z, \overline{a}_2).$$

2.7 Equivalence relation on Z

Let $\mathcal{A} = (Z, A, B, \delta, \lambda)$ be an automaton and $z, z' \in Z$. Then

$$z \sim z' \text{ if } \forall \overline{a} \in \overline{A},\ \overline{\lambda}(z, \overline{a}) = \overline{\lambda}(z', \overline{a}).$$

2.8 Definition (Series composition)

Let $\mathcal{A}_i = (Z_i, A_i, B_i, \delta_i, \lambda_i)$ $(i \in \{1, 2\})$ be automata, with the additional assumption $A_2 = B_1$.

The *series composition* $\mathcal{A}_1 \# \mathcal{A}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as the automaton

$$(Z_1 \times Z_2, A_1, B_2, \delta, \lambda)$$

with

$$\delta((z_1, z_2), a_1) = (\delta_1(z_1, a_1), \delta_2(z_2, \lambda_1(z_1, a_1))),$$
$$\lambda((z_1, z_2), a_1) = \lambda_2(z_2, \lambda_1(z_1, a_1)),$$
$$((z_1, z_2) \in Z_1 \times Z_2, a_1 \in A_1).$$

This automaton operates as follows: An input $a_1 \in A_1$ operates on z_1 and gives a state transition $z_1' = \delta_1(z_1, a_1)$ and an output $b_1 = \lambda_1(z_1, a_2) \in B_1 = A_2$. This output b_1 operates on Z_2, transforms a $z_2 \in Z_2$ into $z_2' = \delta_2(z_2, b_1)$ and produces the output $\lambda_2(z_2, b_1)$. Then $\mathcal{A}_1 \# \mathcal{A}_2$ is in the next state (z_1', z_2').

Analogously, we can define the series composition of an automaton $\mathcal{A} = (Z_1, A_1, B, \delta_1, \lambda)$ and a semiautomaton $\mathcal{S} = (Z_2, B, \delta_2)$ as the semiautomaton

$$\mathcal{A} \# \mathcal{S} = (Z_1 \times Z_2, A_1, \delta),$$

with δ as the previous paragraph.

3 Preferences. Rational choice

Let (t, C) be the expression in monetary unities of a economic good referred to an instant t of the time. Let

$$E = \mathcal{R} \times \mathcal{R}^+ = \{(t, C) / t \in \mathcal{R},\ C \in \mathcal{R}^+\}.$$

We suposse that in E occur the following axioms:

3.1 Axiom 1

In E exists a total preorder relation , e. g., a binary relation $\preceq$, which yields the following properties:

1. *Reflexive*: $\forall (t,C) \in E$, $(t,C) \preceq (t,C)$.
2. *Transitive*: $\forall (t,C)$, (t',C'), $(t'',C'') \in E$,

$$\text{if } (t,C) \preceq (t',C') \text{ and } (t',C') \preceq (t'',C'') \Rightarrow (t,C) \preceq (t'',C'').$$

3. *Complete*$^{(6)}$: $\forall (t,C)$, $(t',C') \in E$,

$$(t,C) \preceq (t',C') \text{ or } (t',C') \preceq (t,C).$$

3.2 Consequences of axiom 1

We consider the relation $\sim$ defined on E as:

$$\forall (t,C),\ (t',C') \in E,$$

$$(t,C) \sim (t',C') \text{ if } (t,C) \preceq (t',C') \text{ and } (t',C') \preceq (t,C).$$

A) $\sim$ is a equivalence relation.

B) $\preceq$ is compatible with $\sim$, in the sense that $(t,C) \preceq (t',C')$, if it verifies $(t,C) \sim (t_1,C_1)$ and $(t',C') \sim (t'_1,C'_1)$ then $(t_1,C_1) \preceq (t'_1,C'_1)$.

C) On $E/\sim$ we define a relation $\preceq$ as:

$$[(t,C)] \preceq [(t',C')] \Leftrightarrow (t,C) \preceq (t',C').$$

We can prove that $\preceq$ is a order relation on $E/\sim$, called *order relation* $\preceq$ *associated with the relation* $\preceq$.

In the other hand, an *utility function* is a function:

$$u : \mathcal{R} \times \mathcal{R}^{+} \longrightarrow \mathcal{R}$$

increasing with preferences. This means that:

$$\text{If } (t,C) \preceq (t',C') \Rightarrow u(t,C) \leq u(t',C').$$

$$\text{If } (t,C) \sim (t',C') \Rightarrow u(t,C) = u(t',C').$$

Therefore, each equivalence class of the relation $\sim$ is

$$[(t,C)] = \{(t',C') \in \mathcal{R} \times \mathcal{R}^{+} / u(t,C) = u(t',C')\}.$$

It is called *indifference curve* and is represented briefly by the level equation $u(t,C) = k$.

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3.3 Axiom 2

$$\text{If } C \leq C' \text{ and } t' \leq t \Rightarrow (t, C) \preceq (t', C').$$

Well now, if $C < C'$ and $t = t'$ or $C = C'$ and $t > t' \Rightarrow (t, C) \prec (t', C')$, where $\prec$ means $\preceq$ but not $=$.

3.4 Consequences of axiom 2

This axiom means that, given a financial capital (t_0, C_0), all points placed on the north-west of (t_0, C_0) are better than (t_0, C_0) and the points placed on the south-east are worse than (t_0, C_0).

Therefore, according to axiom 2, the indifference curves must have a positive slope (> 0) and it is obvious that curves more far of the origin will represent greater *utility index* (k).

3.5 Interpretation of the axiom 2

There can be no doubt that, for each rational economic subject, the measure in monetary unities (C) of a economic good is increasing with regard to the disposibility instant (t) of this good, from what the time can be considered as a "negative economic good".

3.6 Axiom 3

The indifference curves are convex (increasing slope), e. g.,

$$(t_0, C_0) \sim (t_1, C_1) \Rightarrow \alpha(t_0, C_0) + (1 - \alpha)(t_1, C_1) \succeq (t_0, C_0),$$

$$\forall \alpha \in \mathcal{R} / \ 0 \leq \alpha \leq 1.$$

3.7 Interpretation of the axiom 3

See Figure 1.

POINT B ⇒ Very big slope: the economic subjet is prepared to give up a lot of C by changing an unit of t decrease (because the expiration is far according to a fixed point t_0). Said in another way: the interest produced by a quantity C_0 placed in an instant t_0 is big the farther we move from that temporal point.

POINT A ⇒ The slope is minor than B: the subjet is prepared to give up less than C by changing an unit of t (t is minor now).

POINT C ⇒ The slope is still smaller: the subject is prepared to give up very little of C, by changing an unit of t (now the expiration is inmediate, e. g., t is very little).

This axiom could be discussed in practice, as far as interest increases with time, it should not matter if, also, is a relative increase. This information as a known fact, is a consequence of the interest rate$^{(7)}$, that in conclusion is a derivate magnitude. This fact allows a distinction between "financial laws in strict sense" and "financial laws in ample sense".

Financial logic tell us that if a curve is over another one, its slope has to be bigger, to verificate that the interest is increasing in relation to the quantity that is produced, that is:

3.8 Axiom 4

For all $t \in \mathcal{R}$ and for all $a, x \in \mathcal{R}^+$, $C_0 < C_1$ and $(t, C_0) \sim (t+a, C_0+x)$ implies $(t+a, C_1+x) \prec (t, C_1)$.

4 Definition (financial law)

1. We consider the following automaton:

$$\mathcal{A} = (Z, A, FC_{\mathcal{R}}, \delta_1, \lambda_1),$$

where:

a) Z is a subset of $\mathcal{Z}$ (integer numbers set): $Z \subseteq \mathcal{Z}$.

b) A is a subgroup of $(\mathcal{Z}, +)$.

c) FC_B is the set of bijections strictly increasing of a set B, subset of $\mathcal{R}$, onto itself.

d) $\delta_1 : Z \times A \longrightarrow Z/(t,a) \longmapsto \delta_1(t,a) = t+a$ is an action of the group $(A, +)$ on the set Z.

e) $\lambda_1 : Z \times A \longrightarrow FC_B/(t,a) \longmapsto \lambda_1(t,a)$ is a function which verifies the following conditions:

(a) $\forall t \in Z;\ \forall a, a' \in A, \lambda_1(t+a, a') \circ \lambda_1(t,a) = \lambda_1(t, a+a')$.

(b) $\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0, \lambda_1(t,a) < \lambda_1(t, a+x)$, in the sense of a punctual inequality.

(c) $\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0, \lambda_1(t, a+x) - \lambda_1(t,a) \in FC_B$.

(d) $\forall t \in Z;\ \forall a \in A;\ \forall x,\ y \in A^+$,

$$\lambda_1(t, a+x) - \lambda_1(t,a) \leq \lambda_1(t, a+x+y) - \lambda_1(t, a+y).$$

2. And we consider moreover the following semiautomaton:

$$\mathcal{S} = (B, FC_B, \delta_2),$$

where:

a) B is a subset of $\mathcal{R}$: $B \subseteq \mathcal{R}$.

b) $\delta_2 : B \times FC_B \longrightarrow B / (C, f) \mapsto \delta_2(C, f) = f(C)$ is an action of the group $(FC_B, \circ)$ on B.

Well now, a *financial law* is the series composition $\mathcal{L} = \mathcal{A}\#\mathcal{S}$ of $\mathcal{A}$ and $\mathcal{S}$ defined by the semiautomaton

$$(Z \times B, A, \delta),$$

with:

$$\delta((t, C), a) = (\delta_1(t, a), \delta_2(C, \lambda_1(t, a))),$$

e. g.:

$$\delta((t, C), a) = (t + a, \lambda_1(t, a)(C)),$$

4.1 Consequences of financial law definition

1. $\lambda_1(t, 0) = Id_B;\ \forall t \in Z.$

   In fact, $\forall t \in Z;\ \forall a \in A,$

   $$\lambda_1(t, a) \circ \lambda_1(t, 0) = \lambda_1(t, a) \Rightarrow \lambda_1(t, 0) = Id_B.$$

2. $\lambda_1^{-1}(t, a) = \lambda_1(t + a, -a).$

3. $\lambda_1(t, a) < \lambda_1(t, a + x);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0 \Leftrightarrow$
   $\Leftrightarrow \lambda_1(t, 0) < \lambda_1(t, x);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0.$

4. $\lambda_1(t, a) > \lambda_1(t, a + x);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^-,\ x \neq 0 \Leftrightarrow$
   $\Leftrightarrow \lambda_1(t, 0) > \lambda_1(t, x);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^-,\ x \neq 0.$

5. $\lambda_1(t + x, a) < \lambda_1(t, a + x);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0.$

   In fact, because of the first condition of financial law,

   $$\lambda_1(t + x, a) \circ \lambda_1(t, x) = \lambda_1(t, a + x) \Rightarrow$$

   $$\lambda_1(t + x, a) = \lambda_1(t, a + x) \circ \lambda_1(t + x, -x) < \lambda_1(t, a + x).$$

6. $\lambda_1(t + x, a - x) < \lambda_1(t, a);\ \forall t \in Z;\ \forall a \in A;\ \forall x \in A^+,\ x \neq 0.$

   In fact, because of the last consequence,

   $$\lambda_1(t + x, a - x) < \lambda_1(t, a - x + x) = \lambda_1(t, a).$$

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4.2 Financial laws of capitalization and discount

If we restrict δ_1 and λ_1 in the financial law definition to the monoids:

$$A^+ = \{a \in A/a \geq 0\}$$

and

$$A^- = \{a \in A/a \leq 0\},$$

we obtain the concepts of *financial law of capitalization* and *financial law of discount*, respectively.

4.3 Classification of financial laws

a) *According to the set* $Z \times B$:

Let $\mathcal{L} = (Z \times B, A, \delta)$ and $\mathcal{L}' = (Z' \times B', A, \delta')$ be financial laws.

We say that $\mathcal{L} \leq \mathcal{L}'$ if $Z \times B \subseteq Z' \times B'$ and δ and λ are the restrictions of δ' and λ' on $Z \times B$, respectively.

It's about two financial laws that differ only in that the first is applied to some days or some capitals to which the second is applied.

b) *According to the group* A:

We suposse that $Z = \mathbb{Z}$:

1. When $A = \mathbb{Z}$, we say that $\mathcal{L}$ is a financial law associated to a liquid operation.
2. When $A = n\mathbb{Z}$, we say that $\mathcal{L}$ is a financial law associated to an operation of time deposits.
    (a) When $A = 360\mathbb{Z}$, it's about an annual time.
    (b) When $A = 30\mathbb{Z}$, it's about a monthly time.

4.4 Examples

A) $A = Z = \mathbb{Z}$; $B = \mathcal{R}^+$; $\lambda_1(t, a) = Id_B.e^{ka}$, $k > 0$.

B) $A = Z = \mathbb{Z}$; $B =]1, +\infty[$; $\lambda_1(t, a) = (Id_B)^{k^a}$, $k > 1$.

The transformed of five quantities by the action of sixteen inputs, when $k = 1{,}01$, are:

See Table 1.

The profitabilities, using the compound capitalization, are:

See Table 2.

As we can observe, the profitabilities increase in accordance with the quantity and the time.

C) $A = Z = \mathcal{Z}^+;\ B = \mathcal{R}^+;\ \lambda_1(t,a) = Id_B.\frac{t+a+k}{t+k}, k > 0.$

D) $A = \alpha\mathcal{Z};\ Z = \mathcal{Z};\ B = \mathcal{R}^+;\ \lambda_1(t,a) = Id_B.(1+\alpha i)^{\frac{a}{\alpha}},\ i > 0.$

E) $A = 30\mathcal{Z}; Z = \mathcal{Z};\ B =]1,+\infty[;\ \lambda_1(t,30)(C) = C.(1+0'003.log C).$

The transformed of five quantities by the action of twelve inputs are:
See Table 3.
The profitabilities, using the simple capitalization, are:
See Table 4.

As we can observe, the profitabilities increase in accordance with the quantity and the time.

4.5 Graphic representation of a financial law

See Figure 2.

4.6 Extension of the financial law concept

We extend the input set A to the free group $\overline{A} = \overline{F}_A$ and FC_B to the free group $\overline{FC}_B$, with Λ as identity.

We also extend δ_1 and λ_1 from $Z \times A$ to $Z \times \overline{A}$ defining for $t \in Z$ and $a_1, a_2, \ldots\ldots, a_r \in A$:

$$\overline{\delta}_1(t,\Lambda) = t,$$

$$\overline{\delta}_1(t,a_1) = \delta_1(t,a_1) = t + a_1,$$

$$\overline{\delta}_1(t,a_1a_2) = \delta_1(\overline{\delta}_1(t,a_1),a_2) =$$

$$= \delta_1(t + a_1, a_2) = (t + a_1) + a_2 = t + (a_1 + a_2),$$

$$\vdots$$

$$\overline{\delta}_1(t,a_1a_2\ldots\ldots a_r) = \delta_1(\overline{\delta}_1(t,a_1a_2\ldots\ldots a_{r-1}),a_r) = t + (a_1 + a_2 + \ldots\ldots + a_r),$$

and

$$\overline{\lambda}_1(t,\Lambda) = \Lambda,$$

$$\overline{\lambda}_1(t,a_1) = \lambda_1(t,a_1),$$

$$\overline{\lambda}_1(t,a_1a_2) = \lambda_1(t,a_1)\overline{\lambda}_1(\delta_1(t,a_1),a_2) = \lambda_1(t,a_1)\lambda_1(t+a_1,a_2),$$

$$\vdots$$

$$\overline{\lambda}_1(t,a_1a_2\ldots\ldots a_r) = \lambda_1(t,a_1)\overline{\lambda}_1(\delta_1(t,a_1),a_2a_3\ldots\ldots a_r) =$$

$$= \lambda_1(t,a_1)\overline{\lambda}_1(t+a_1,a_2a_3\ldots\ldots a_r) =$$

$$= \lambda_1(t,a_1)\lambda_1(t+a_1,a_2)\ldots\ldots\lambda_1(t+a_1+a_2+\ldots\ldots+a_{r-1},a_r).$$

Moreover, we extend δ_2 from $B \times FC_B$ to $B \times \overline{FC}_B$, defining for $C \in B$ and $f_1, f_2, \ldots\ldots, f_r \in FC_B$:

$$\overline{\delta}_2(C, \Lambda) = C,$$

$$\overline{\delta}_2(C, f_1) = \delta_2(C, f_1) = f_1(C),$$

$$\overline{\delta}_2(C, f_1 f_2) = \delta_2(\overline{\delta}_2(C, f_1), f_2) =$$

$$= \delta_2(f_1(C), f_2) = f_2[f_1(C)] = (f_2 \circ f_1)(C),$$

$$\vdots$$

$$\overline{\delta}_2(C, f_1 f_2 \ldots\ldots f_r) = \delta_2(\overline{\delta}_2(C, f_1 f_2 \ldots\ldots f_{r-1}), f_r) = (f_r \circ \ldots\ldots \circ f_2 \circ f_1)(C).$$

4.7 Particular case

If $f_1, f_2, \ldots\ldots, f_r$ are homogeneous of degree one with regard to C:

$$\overline{\delta}_2(C, f_1 f_2 \ldots\ldots f_r) = (f_r \circ \ldots\ldots \circ f_2 \circ f_1)(C) = C.f_1(1).f_2(1)\ldots\ldots f_r(1).$$

In this case, the financial law $\mathcal{L}$ (homogeneous of degree one with regard to the quantity) is called*classical financial law*.

4.8 Graphic representation

Let

$$(t_1, C_1) = \delta((t, C), a_1),$$

$$(t_2, C_2) = \overline{\delta}((t, C), a_1 a_2) = \overline{\delta}(\delta((t, C), a_1), a_2),$$

$$\vdots$$

If the semiautomaton is in the capital (t, C) and an input sequence $a_1 a_2 \ldots\ldots a_r \in \overline{A}$ operates, then the capitals are changed from (t, C) to (t_1, C_1), until the final capital (t_r, C_r) is obtained. See Figure 3.

In the following, $\equiv_1$ (resp. $\equiv_2$) is the equivalence relation on $\overline{A}$ (resp. $\overline{FC}_B$) associated to the semiautomaton (Z, A, δ_1) (resp. (B, FC_B, δ_2)) (see section 2.6). Moreover, $\sim_1$ is the equivalence relation on Z associated to the automaton $(Z, A, FC_B, \delta_1, \lambda_1)$ (see section 2.7).

5 Algebraic properties

5.1 Theorem

Let $\mathcal{L}$ be a financial law. Then $M_{\mathcal{L}}$ is an abelian group and $A \cong M_{\mathcal{L}}$ as groups.

Proof. We know that $M_{\mathcal{L}}$ is a monoid. Well now, $M_{\mathcal{L}}$ is a group ?. We consider any element $f_{\overline{a}} \in M_{\mathcal{L}}$. If $\overline{a} = a_1 a_2 .....a_n$, let

$$-\overline{a} = (-a_1)(-a_2).....(-a_n).$$

Then it verifies that

$$f_{\overline{a}} \circ f_{-\overline{a}} = f_{-\overline{a}} \circ f_{\overline{a}} = Id_{Z \times B}.$$

Analogously, it proves that:

$$f_{-\overline{a}} \circ f_{\overline{a}} = Id_{Z \times B}.$$

Therefore, $M_{\mathcal{L}}$ is a group. Moreover, $M_{\mathcal{L}}$ is an abelian group.

In the other hand, we establish the followig correspondence:

$$\varphi : \overline{A}/ \equiv \longrightarrow M_{\mathcal{L}}$$

such that:

$$[\overline{a}] \mapsto \varphi([\overline{a}]) = f_{\overline{a}}.$$

φ es bijective (Lidl and Pilz (1984)). Moreover φ *is an homomorphism of groups.*

At last, as

$$\overline{A}/ \equiv_1 \cong A,$$

$$\overline{A}/ \equiv \text{ coincides with } A/ \equiv_1$$

and

$$\overline{A}/ \equiv \cong M_{\mathcal{L}},$$

it verifies that:

$$A \cong M_{\mathcal{L}}.$$

A semigroup S_1 *divides* a semigroup S_2, if S_1 is a homomorphic image of a subsemigroup of S_2. In symbols: $S_1 | S_2$.

Let $\mathcal{A}_1 = (Z_1, A_1, B_1, \delta_1, \lambda_1)$ and $\mathcal{A}_2 = (Z_2, A_2, B_2, \delta_2, \lambda_2)$ be automata. An *(automata-) homomorphism* $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a triple $\Phi = (\zeta, \alpha, \beta)$, $\zeta : Z_1 \longrightarrow Z_2$, $\alpha : A_1 \longrightarrow A_2$, $\beta : B_1 \longrightarrow B_2$ with the property

$$\zeta(\delta_1(z, a)) = \delta_2(\zeta(z), \alpha(a)),$$

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$$\beta(\delta_1(z,a)) = \lambda_2(\zeta(z), \alpha(a)).$$

Φ is called a *monomorphism* (*epimorphism, isomorphism*) if all functions ζ, α and β are injective (surjective, bijective).

An automaton $\mathcal{A}_1 = (Z_1, A, B, \delta_1, \lambda_1)$ (resp., a semiautomaton $\mathcal{S}_1 = (Z_1, A, \delta_1)$) *divides* an automaton $\mathcal{A}_2 = (Z_2, A, B, \delta_2, \lambda_2)$ (resp., a semiautomaton $\mathcal{S}_2 = (Z_2, A, \delta_2)$) (equal input and output alphabets) if $\mathcal{A}_1$ (resp., $\mathcal{S}_1$) is a homomorphic image of a subautomaton of $\mathcal{A}_2$ (resp., of a subsemiautomaton of $\mathcal{S}_2$). In symbols: $\mathcal{A}_1|\mathcal{A}_2$ (resp., $\mathcal{S}_1|\mathcal{S}_2$).

Two semigroups, automata or semiautomata are called *equivalent*, if they divide each other. In symbols: $S_1 \sim S_2$, $\mathcal{A}_1 \sim \mathcal{A}_2$ or $\mathcal{S}_1 \sim \mathcal{S}_2$.

5.2 Theorem

1. Isomorphic automata (resp., semiautomata) are equivalent (but not conversely).
2. $\mathcal{A}_1|\mathcal{A}_2 \Leftrightarrow M_{\mathcal{A}_1}|M_{\mathcal{A}_2}$.
3. $\mathcal{A}_1 \sim \mathcal{A}_2 \Leftrightarrow M_{\mathcal{A}_1} \sim M_{\mathcal{A}_2}$.

Proof. See Lidl and Pilz (1984), p. 368.

On the following, we plan the problem of an scale change in the quantity.

5.3 Theorem

Let $\mathcal{A}_1 = (Z_1, A, B, \delta_1, \lambda_1)$ be an automaton. We suposse that the set Z_1 is isomorphic to the set Z_2, e. g., $\exists f : Z_1 \longrightarrow Z_2$ bijective.

We define:

1.

$$\delta_2 : Z_2 \times A \longrightarrow Z_2 /$$
$$(z_2, a) \mapsto \delta_2(z_2, a) = f[\delta_1(f^{-1}(z_2), a)] :$$

$$\begin{tikzcd}
Z_2 \times A \arrow[r, "\delta_2"] \arrow[d, "f^{-1} \times Id_A"'] & Z_2 \\
Z_1 \times A \arrow[r, "\delta_1"'] & Z_1 \arrow[u, "f"']
\end{tikzcd}$$

2.

$$\lambda_2 : Z_2 \times A \longrightarrow B /$$

$$(z_2, a) \mapsto \lambda_2(z_2, a) = \lambda_2(f^{-1}(z_2), a) :$$

$$\begin{tikzcd}
Z_2 \times A \arrow[r, "\lambda_2"] \arrow[d, "f^{-1} \times Id_A"'] \arrow[dr, phantom, "\cdot"] & B \\
Z_1 \times A \arrow[ur, "\lambda_1"'] &
\end{tikzcd}$$

Then it verifies that $\mathcal{A}_1 \sim \mathcal{A}_2$, being:

$$\mathcal{A}_2 = (Z_2, A, B, \delta_2, \lambda_2).$$

Proof. In fact, we define:

$$\varphi : M_{\mathcal{A}_1} \longrightarrow M_{\mathcal{A}_2} /$$

$$(f^{(1)}_{\overline{a}} : Z_1 \longrightarrow Z_1) \mapsto \varphi(f^{(1)}_{\overline{a}}) = (f^{(2)}_{\overline{a}} : Z_2 \longrightarrow Z_2).$$

φ is an isomorphism of semigroups.

1. *φ is a mapping*: Let us suppose that $f^{(1)}_{\overline{a}} = f^{(1)}_{\overline{a}'}$:

$$\forall z_1 \in Z_1,\ f^{(1)}_{\overline{a}}(z_1) = f^{(1)}_{\overline{a}'}(z_1) \Rightarrow \overline{\delta}_1(z_1, \overline{a}) = \overline{\delta}_1(z_1, \overline{a}').$$

$$\forall z_2 \in Z_2,\ f^{(2)}_{\overline{a}}(z_2) = \overline{\delta}_2(z_2, \overline{a}) = f[\overline{\delta}_1(f^{-1}(z_2), \overline{a})] =$$

$$= f[\overline{\delta}_1(f^{-1}(z_2), \overline{a}')] = \overline{\delta}_2(z_2, \overline{a}') = f^{(2)}_{\overline{a}'}(z_2) \Rightarrow \varphi(f^{(1)}_{\overline{a}}) = \varphi(f^{(1)}_{\overline{a}'}).$$

2. *φ is injective*: Let us suppose that $\varphi(f^{(1)}_{\overline{a}}) = \varphi(f^{(1)}_{\overline{a}'})$:

$$\forall z_2 \in Z_2,\ f^{(2)}_{\overline{a}}(z_2) = f^{(2)}_{\overline{a}'}(z_2) \Rightarrow \overline{\delta}_2(z_2, \overline{a}) = \overline{\delta}_2(z_2, \overline{a}').$$

$$\forall z_1 \in Z_1,\ f[f^{(1)}_{\overline{a}}(z_1)] = f[\overline{\delta}_1(z_1, \overline{a})] = f[\overline{\delta}_1(f^{-1}(z_2), \overline{a})], \text{ for some } z_2 \in Z_2 =$$

$$= \overline{\delta}_2(z_2, \overline{a}) = \overline{\delta}_2(z_2, \overline{a}') = f[\overline{\delta}_1(f^{-1}(z_2), \overline{a}')] = f[\overline{\delta}_1(z_1, \overline{a}')] =$$

$$= f[f^{(1)}_{\overline{a}'}(z_1)] \Rightarrow \forall z_1 \in Z_1, f[f^{(1)}_{\overline{a}}(z_1)] = f[f^{(1)}_{\overline{a}'}(z_1)].$$

As f is injective, $f^{(1)}_{\overline{a}}(z_1) = f^{(1)}_{\overline{a}'}(z_1) \Rightarrow f^{(1)}_{\overline{a}} = f^{(1)}_{\overline{a}'}$.

3. *φ is surjective*:

$$\forall f^{(2)}_{\overline{a}} \in M_{\mathcal{A}_2},\ \exists f^{(1)}_{\overline{a}} \in M_{\mathcal{A}_1} / \varphi(f^{(1)}_{\overline{a}}) = f^{(2)}_{\overline{a}}.$$

It is obvious.

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4. *φ is an homomorphism of semigroups:*

$$\forall f^{(1)}_{\bar{a}}, f^{(1)}_{\bar{a}'} \in M_{\mathcal{A}_1},\ \varphi(f^{(1)}_{\bar{a}} \circ f^{(1)}_{\bar{a}'}) =$$

$$= \varphi(f^{(1)}_{\bar{a}'\bar{a}}) = f^{(2)}_{\bar{a}'\bar{a}} = f^{(2)}_{\bar{a}} \circ f^{(2)}_{\bar{a}'} = \varphi(f^{(1)}_{\bar{a}}) \circ \varphi(f^{(1)}_{\bar{a}'}).$$

Therefore,

$$M_{\mathcal{A}_1} \sim M_{\mathcal{A}_2} \Leftrightarrow \mathcal{A}_1 \sim \mathcal{A}_2.$$

5.4 Corollary

Let $\mathcal{L} = (Z \times B, A, \delta)$ be a financial law of which the set B of quantities is expressed in a monetary unit. Let B_m be the set of the previous quantities expressed in another monetary unit m. In these conditions, exists a financial law

$$\mathcal{L}_m = (Z \times B_m, A, \delta_m)$$

which is equivalent to $\mathcal{L}$.

In this way, every financial law $\mathcal{L}$ will be applied to any money: peseta, dolar, mark, etc..

In fact, if we use as a measure unit the money m, given a financial law $\mathcal{L}$, we will have a bijection:

$$f_m : B \longrightarrow B_m,$$

from which we will obtain a financial law $\mathcal{L}_m$, such that $\mathcal{L}_m \sim \mathcal{L}$.

Analogously, last theorem let us work on thousands, millions, etc. of that money, given a financial law in some money m, as this transformation represents an scale change, the same as the money changes too.

5.5 Preference relation between financial capitals

Let $\mathcal{L}$ be a financial law. On $Z \times B$ we define the following binary relation, in an extensive sense:

$$(t, C) \preceq (t', C') \Leftrightarrow t' - t \in A \text{ and } \lambda_1(t, t' - t)(C) \leq C'.$$

The indifference relation associated to the preference relation is the following:

5.6 Equivalence between financial capitals

Let $\mathcal{L}$ be a financial law. $(t, C) \sim (t', C') \Leftrightarrow \exists a \in A / \delta((t, C), a) = (t', C')$.

Well now,

$$\delta((t, C), a) = (t', C')$$

and

$$\delta((t,C),a) = (\delta_1(t,a), \delta_2(C, \lambda_1(t,a))),$$

from what

$$(\delta_1(t,a), \delta_2(C, \lambda_1(t,a))) = (t', C'),$$

or, analogously:

1.
$$\delta_1(t,a) = t' \Leftrightarrow t + a = t' \Leftrightarrow a = t' - t.$$

2.
$$\delta_2(C, \lambda_1(t,a)) = C' \Leftrightarrow \lambda_1(t,a)(C) = C'.$$

As might have been expected, the binary relation defined before is an equivalence relation, as it affirms subsequently:

5.7 Theorem

$\sim$ is an equivalence relation.

Obviously, this equivalence relation, in the set $(t^* + A) \times B$, where t^* is any element of Z, verifies the axioms 1, 2, 3 and 4 (see section 3).

The conditions of financial law definition (or axioms 1, 2, 3 and 4 in section 3) are the following interpretation in an space with a discreet horizontal dimension:

5.8 Geometric interpretation

Condition 1: A financial capital can "move freely" through an indifference curve.

Condition 2: The indifference curves are strictly increasing.

Condition 3: If an indifference curve is over another one, its slope is bigger.

Condition 4: The indifference curves are concave (strictly or not).

5.9 Meaning of relations $\equiv$ and $\sim$ in a financial law $\mathcal{L}$

A) *What is the meaning of* $\equiv_1$ *?:*

We have:

$$\overline{a} \equiv_1 \overline{a}' \Leftrightarrow f_{\overline{a}} = f_{\overline{a}'} \Leftrightarrow \forall t \in Z,\ f_{\overline{a}}(t) = f_{\overline{a}'}(t) \Leftrightarrow$$

$$\Leftrightarrow \forall t \in Z,\ \overline{\delta}_1(t,\overline{a}) = \overline{\delta}_1(t,\overline{a}') \Leftrightarrow (\text{if } \overline{a} = a_1a_2.....a_n \text{ and } \overline{a}' = a'_1a'_2.....a'_m) \Leftrightarrow$$

$$\Leftrightarrow t + a_1 + a_2 + ... + a_n = t + a'_1 + a'_2 + ... + a'_m \Leftrightarrow$$

$$\Leftrightarrow a_1 + a_2 + ... + a_n = a'_1 + a'_2 + ... + a'_m.$$

We define the following mapping:

$$\varphi : \overline{A}/\equiv_1 \longrightarrow A,$$

defined by:

$$[\overline{a}] = [a_1 a_2 .....a_n] \mapsto \varphi([\overline{a}]) = a_1 + a_2 + ... + a_n.$$

1. The previous paragraph proves that φ is well defined.
2. φ is injective, because, in the previous paragraph, it verifies the equivalence.
3. φ is surjective, because $\forall a \in A, \varphi([a]) = a$. Therefore, φ is bijective.
4. φ is an homomorphism of groups:

$$\varphi : (\overline{A}/\equiv_1, \text{concatenation}) \longrightarrow (A, +),$$

being

$$[\overline{a}][\overline{a}'] = [\overline{a}\overline{a}'].$$

In fact,

$$\varphi([\overline{a}][\overline{a}']) = \varphi([\overline{a}\overline{a}']) = \varphi([\overline{a}]) + \varphi([\overline{a}']).$$

Therefore, $(\overline{A}/\equiv_1, \text{concatenation}) \cong (A, +)$.

B) *What is the meaning of* $\sim_1$ *?:*

We have:

$$t \sim_1 t' \Leftrightarrow \forall \overline{a} \in \overline{A} : \overline{\lambda}_1(t, \overline{a}) = \overline{\lambda}_1(t', \overline{a}) \Leftrightarrow (\text{if } \overline{a} = a_1 a_2 .....a_n) \Leftrightarrow$$

$$\Leftrightarrow \lambda_1(t, a_1)\lambda_1(t + a_1, a_2).....\lambda_1(t + a_1 + ... + a_{n-1}, a_n) =$$

$$= \lambda_1(t', a_1)\lambda_1(t' + a_1, a_2).....\lambda_1(t' + a_1 + ... + a_{n-1}, a_n) \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{rcl} \lambda_1(t, a_1) & = & \lambda_1(t', a_1) \\ \lambda_1(t + a_1, a_2) & = & \lambda_1(t' + a_1, a_2) \\ \vdots & = & \vdots \\ \lambda_1(t + a_1 + ... + a_{n-1}, a_n) & = & \lambda_1(t' + a_1 + ... + a_{n-1}, a_n) \end{array} \right.$$

$$\Leftrightarrow t + a \sim_1 t' + a, \ \forall a \in A.$$

This allows to define the following automaton:

$$\delta_1^* : Z/\sim_1 \times A \longrightarrow Z/\sim_1$$

where:

$$([t], a) \mapsto \delta_1^*([t], a) = [t + a].$$

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The previous paragraph proves that δ_1^* is well defined.

$$\lambda_1^* : Z/\sim_1 \times A \longrightarrow FC_B$$

where:

$$([t], a) \mapsto \lambda_1^*([t], a) = \lambda_1(t, a).$$

$$\delta_2^* : B \times FC_B \longrightarrow B$$

defined by:

$$(C, f) \mapsto \delta_2^*(C, f) = f(C).$$

The semiautomaton $\mathcal{A}^*\#\mathcal{S}^*$, being $\mathcal{A}^* = (Z/\sim_1, A, FC_B, \delta_1^*, \lambda_1^*)$ and $\mathcal{S}^* = (B, FC_B, \delta_2^*)$ is called the equivalent minimal semiautomaton of $\mathcal{A}$.

More particularly,

$$\mathcal{L}^* = \mathcal{A}^*\#\mathcal{S}^* = (Z/\sim_1 \times B, A, \delta^*)$$

is called the *equivalent minimal financial law of* $\mathcal{A}$.

6 Stationary financial laws

The previous construction justifies the following definition

6.1 Definition (Stationary financial law)

A financial law is called an *stationary financial law* if $Card(Z/\sim_1) = 1$, e. g., $Z/\sim_1 = \{[t_0]\}$.

In this case, δ^* and λ^* only depend of A and B.

More generally, the previous equivalence relation $\sim_1$ allows the following

6.2 Clasification of financial laws

1. *Stationary or stationary financial laws of order 1* if $Card(Z/\sim_1) = 1$, e. g., $Z/\sim_1 = \{[t_0]\}$ or, analogously,

$$\forall t \in Z,\ \forall \overline{a} \in \overline{A},\ \overline{\lambda}_1(t, \overline{a}) = \overline{\lambda}_1(t_0, \overline{a}).$$

2. *Stationary financial laws of order n* if $Card(Z/\sim_1) = n$, e. g., $Z/\sim_1 = \{[t_1], [t_2], \ldots.., [t_n]\}$ or, analogously,

$$\forall t \in Z,\ \forall \overline{a} \in \overline{A},\ \exists t_i (i = 1, 2, \ldots.., n) / \overline{\lambda}_1(t, \overline{a}) = \overline{\lambda}_1(t_i, \overline{a}).$$

3. *Dynamic financial laws* if $Card(Z/\sim_1) = Card(Z)$, e. g., $Z/\sim_1 = Z$ or, analogously,

$$\forall t, t' \in Z,\ t \sim_1 t' \Rightarrow t = t'.$$

6.3 Examples

A) Let us consider the following financial law in which $Z = A = \mathcal{Z}$ and $B = \mathcal{R}^+$:

$$\delta_1 : \mathcal{Z} \times \mathcal{Z} \longrightarrow \mathcal{Z}$$

defined by:

$$(t,a) \mapsto \delta_1(t,a) = t + a,$$

$$\lambda_1 : \mathcal{Z} \times \mathcal{Z} \longrightarrow FC_B$$

such that:

$$(t,a) \mapsto \lambda_1(t,a) : \mathcal{R}^+ \to \mathcal{R}$$

defined, at a time, by:

$$\forall C \in B,\ \lambda_1(t,a)(C) = C.e^{ka}$$

being $k > 0$.

In this case,

$$\forall t \in Z,\ \forall \overline{a} \in \overline{A},\ \overline{\lambda}_1(t,\overline{a}) = \overline{\lambda}_1(t_0,\overline{a}) \Rightarrow$$

$$\Rightarrow Z/\sim_1 = \{[t_0]\} \Rightarrow \mathcal{L} \text{ is stationary or stationary of order 1.}$$

B) Let us consider the following financial law in which $Z = \mathcal{Z}$, $A = 2\mathcal{Z}$ and $B = \mathcal{R}^+$:

$$\delta_1 : \mathcal{Z} \times 2\mathcal{Z} \longrightarrow \mathcal{Z}$$

defined by:

$$(t,a) \mapsto \delta_1(t,a) = t + a,$$

$$\lambda_1 : \mathcal{Z} \times 2\mathcal{Z} \longrightarrow FC_B$$

such that:

$$(t,a) \mapsto \lambda_1(t,a) : \mathcal{R}^+ \to \mathcal{R}$$

defined, at a time, by:

$$\forall C \in B,\ \lambda_1(t,a)(C) = C.e^{ka(|sin\frac{\pi}{2}(t-1)|+1)},$$

being $k > 0$, or, analogously:

$$\lambda_1(t,a)(C) = \begin{cases} C.e^{ka}, & if\ t\ is\ odd \\ C.e^{2ka}, & if\ t\ is\ even \end{cases}$$

because:

$$|\ sin\frac{\pi}{2}(t-1)\ | = \begin{cases} 0, & if\ t\ is\ odd \\ 1, & if\ t\ is\ even \end{cases}$$

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In this case,

$$Z/\sim_1 = \{[t_0], [t_0+1]\} \Rightarrow \mathcal{L} \text{ is stationary of order } 2.$$

C) Generally, if

$$Z = \mathcal{Z},\ A = n\mathcal{Z},\ B = \mathcal{R}^+$$

and

$$\lambda_1(t,a)(C) = C.e^{ka(|sin\frac{(n-1)\pi}{n}(t-1)|+1)},\ k > 0,$$

it verifies that

$$Z/\sim_1 = \{[t_0], [t_0+1], \ldots\ldots, [t_0+n-1]\} \Rightarrow \mathcal{L} \text{ is stationary of order } n.$$

D) Let us consider the following financial law in which $Z = A = \mathcal{Z}$ and $B = \mathcal{R}^+$:

$$\delta_1 : \mathcal{Z} \times \mathcal{Z} \longrightarrow \mathcal{Z}$$

defined by:

$$(t,a) \mapsto \delta_1(t,a) = t + a,$$

$$\lambda_1 : \mathcal{Z} \times \mathcal{Z} \longrightarrow FC_B$$

such that:

$$(t,a) \mapsto \lambda_1(t,a) : \mathcal{R}^+ \to \mathcal{R}$$

defined, at a time, by:

$$\forall C \in B,\ \lambda_1(t,a)(C) = C.\frac{t+a+k}{t+k},$$

being $k > 0$.

We suppose that

$$t \sim_1 t' \Rightarrow \forall \overline{a} \in \overline{A},\ \lambda_1(t,\overline{a}) = \lambda_1(t',\overline{a}).$$

Thus,

$$\forall a \in A,\ \forall C \in B,\ \lambda_1(t,a)(C) = \lambda_1(t',a)(C) \Rightarrow$$

$$\Rightarrow C.\frac{t+a+k}{t+k} = C.\frac{t'+a+k}{t'+k} \Rightarrow t' = t.$$

Therefore, $\mathcal{L}$ is dynamic.

E) Let us consider the following financial law in which $Z = \mathcal{Z}$, $A = n\mathcal{Z}$ and $B = \mathcal{R}^+$:

$$\delta_1 : \mathcal{Z} \times n\mathcal{Z} \longrightarrow \mathcal{Z}$$

defined by:

$$(t,a) \mapsto \delta_1(t,a) = t + a,$$

$$\lambda_1 : \mathcal{Z} \times n\mathcal{Z} \longrightarrow FC_B$$

such that:

$$(t,a) \mapsto \lambda_1(t,a) : \mathcal{R}^+ \to \mathcal{R}$$

defined, at a time, by:

$$\forall C \in B,\ \lambda_1(t,a)(C) = C.e^{ka[t]_n+1},$$

being $k > 0$, $n \in \mathcal{N}^*$ and $[t]_n$ the remainder of t divided by n.

Thus:

$$\lambda_1(t,a)(C) = \begin{cases} C.e^{ka.0}, & if\ t = \dot{n}, \\ C.e^{ka.1}, & if\ t = \dot{n}+1, \\ \vdots & \vdots \\ C.e^{ka.(n-1)}, & if\ t = \dot{n}+n-1. \end{cases}$$

In this case,

$$Z/\sim_1 = \{[t_0],[t_0+1],.....,[t_0+n-1]\} \Rightarrow \mathcal{L} \text{ is stationary of order } n.$$

In this way, to relacionate the concepts of stationary law of order n, we can enunciate the following

6.4 Theorem

Let us suppose that $A = n\mathcal{Z}$. In these conditions, $\mathcal{L} = (Z \times B, A, \delta)$ is an stationary financial law of order n if and only if exists n stationary financial laws

$$\mathcal{L}_1, \mathcal{L}_2, ...,\mathcal{L}_n,$$

being

$$\begin{aligned} \mathcal{L}_1 &= (Z_1 \times B, A, \delta_1), \\ \mathcal{L}_2 &= (Z_2 \times B, A, \delta_2), \\ &\vdots \\ \mathcal{L}_n &= (Z_n \times B, A, \delta_n), \end{aligned}$$

such that

1. Z is the disjoint union of $\{Z_i;\ i = 1,.....,n\}$.
2. $\mathcal{L}_1, \mathcal{L}_2, ...,\mathcal{L}_n \leq \mathcal{L}$.

Proof. i) $\Rightarrow$) Let $\mathcal{L} = (Z \times B, A, \delta)$ be an stationary financial law of order n $\Rightarrow Z/\sim_1 = \{[t_1],[t_2],.....,[t_n]\}$. If we denote $Z_1 = [t_1], Z_2 = [t_2],.....,Z_n = [t_n]$, Z is the disjoint union of $Z_i, i = 1,.....,n$.

If δ_i is the restriction of δ to $Z_i \times B$, then $\mathcal{L}_1, \mathcal{L}_2,.....,\mathcal{L}_n \leq \mathcal{L}$. Obviously, $\mathcal{L}_i$ is an stationary financial law.

ii) $\Leftarrow$ It is obvious, because, in this case, $Z/\sim_1 = \{Z_1, Z_2, ...,Z_n\}$.

6.5 Interpretation, from a financial point of view, of the stationary laws of order n

The stationary financial laws of order n musn't be considerated as a pure abstraction because it could be thought in a financial entity that using a stationary financial law of order 30 in which:

$$Z = \mathcal{Z},\ A = 30\mathcal{Z}$$

and

$$\lambda_i(t_i, 30) < \lambda_j(t_j, 30),\ \text{for } i < j,$$

at which it would remunerate plus a deposit in the space of a month made on the first of each month if that same quantity was invested on the second day, and so on, with the objet of raising the colocation in the space of capital volumes important punctually, as, e. g., the payrolls cashed at the begin of the month or the treasurership excedents in determinated dates.

On the following, we are going to establish the first algebraic characterization of the concept of stationary financial law.

Let $\mathcal{L}$ be a financial law and let us suppose that $(Z, +)$ is a subgroup of $(\mathcal{Z}, +)$. In these conditions, the following is verified

6.6 First theorem of characterization

$\mathcal{L}$ is stationary if and only if λ_1 is an homomorphism of the groups $(Z \times A, +)$ and $(FC_B, \circ)$, e. g.:

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t,a) \circ \lambda_1(t',a') \Leftrightarrow$$

$$\Leftrightarrow \lambda_1(t+t', a+a') = \lambda_1(t,a) \circ \lambda_1(t',a').$$

Proof. i) $\Rightarrow$) First let us suppose that the financial law $\mathcal{L}$ is stationary. We will prove that λ_1 is an homomorphism of groups, e. g., that

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t,a) \circ \lambda_1(t',a'), \forall t,t' \in Z;\ \forall a,a' \in A.$$

In fact,

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t+t', a+a') =$$

$$= (\text{because of the first condition of financial law}) =$$

$$= \lambda_1(t+t'+a', a) \circ \lambda_1(t+t', a') =$$

$$= (\text{as } \mathcal{L} \text{ is a stationary law}) = \lambda_1(t,a) \circ \lambda_1(t',a').$$

ii) $\Leftarrow$) Now let us suppose that λ_1 is an homomorphism of groups. We will prove that $\mathcal{L}$ is stationary, e. g., that $\overline{\lambda}_1(t,\overline{a}) = \overline{\lambda}_1(t',\overline{a})$.

In fact,

$$\overline{\lambda}_1(t,\overline{a}) = (\text{if } \overline{a} = a_1a_2.....a_n) =$$

$$= \lambda_1(t,a_1)\lambda_1(t+a_1,a_2).....\lambda_1(t+a_1+.....+a_{n-1},a_n) =$$

$$= \lambda_1((t,0)+(0,a_1))\lambda_1((t+a_1,0)+(0,a_2)).....$$

$$.....\lambda_1((t+a_1+.....+a_{n-1},0)+(0,a_n)) =$$

$$= (\lambda_1(t,0)\circ\lambda_1(0,a_1))(\lambda_1(t+a_1,0)\circ\lambda_1(0,a_2)).....$$

$$.....(\lambda_1(t+a_1+.....+a_{n-1},0)\circ\lambda_1(0,a_n)) =$$

$$= (\lambda_1(t',0)\circ\lambda_1(0,a_1))(\lambda_1(t'+a_1,0)\circ\lambda_1(0,a_2)).....$$

$$.....(\lambda_1(t'+a_1+.....+a_{n-1},0)\circ\lambda_1(0,a_n)) =$$

$$= \overline{\lambda}_1(t',a_1a_2.....a_n) = \overline{\lambda}_1(t',\overline{a}) \Rightarrow \overline{\lambda}_1(t,\overline{a}) = \overline{\lambda}_1(t',\overline{a}).$$

Therefore, $\mathcal{L}$ is stationary.

On the following, let us expose the second characterization of the concept of stationary financial law.

Let $\mathcal{L}$ be a financial law and let us suppose that $(Z,+)$ is a subgroup of $(\mathcal{Z},+)$. In these conditions, the following is verified

6.7 Second theorem of characterization

$\mathcal{L}$ is stationary if and only if

$$(\{\lambda_1(t,a)/t \in Z, a \in A\},\circ)$$

is a cyclic group such that

$$\lambda_1(t,a) = [\lambda_1(0,\alpha)]^{\frac{a}{\alpha}},\ \forall t \in Z,\ \forall a \in A,$$

being α the generator of A.

Proof. i) $\Rightarrow$ Let us suppose $\mathcal{L}$ is a stationary financial law. Then it verifies that

$$\forall t \in Z,\ \forall a \in A,\ \lambda_1(t,a) = \lambda_1(t,\alpha+\alpha+.....\overset{\frac{a}{\alpha}\ times}{\underbrace{\qquad\qquad}}.....+\alpha) =$$

$$= \lambda_1(t+\alpha+.....+\alpha,\alpha)\circ.....\circ\lambda_1(t+\alpha,\alpha)\circ\lambda_1(t,\alpha) =$$

$$= (\text{as } \mathcal{L} \text{ is a stationary financial law}) =$$

$$= \lambda_1(0,\alpha)\circ.....\overset{\frac{a}{\alpha}\ times}{\underbrace{\qquad\qquad}}.....\circ\lambda_1(0,\alpha)\circ\lambda_1(0,\alpha) = [\lambda_1(0,\alpha)]^{\frac{a}{\alpha}},$$

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being α the generator of A. This reasoning is valid when a and α are equal sign. In other case, we use the following equality:

$$\lambda_1(t,a) = \lambda_1(t, -\alpha - \alpha - \ldots\ldots \overset{-\frac{a}{\alpha}\ times}{\underbrace{\qquad\qquad}} \ldots\ldots - \alpha).$$

Therefore, $(\{\lambda_1(t,a)/t \in Z, a \in A\}, \circ)$ is a cyclic group.

A consequence of the last theorem is the general expression of the stationary financial laws, as it proves in the following

6.8 Corollary

The expression of a stationary financial law is:

$$\lambda_1(t,a) = K^{\frac{a}{\alpha}},$$

with $K > Id_B$.

6.9 Example

Let us consider the following financial law in which $Z = A = \mathbb{Z}$ and $B =]1, +\infty[$:

$$\delta_1 : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

defined by:

$$(t,a) \mapsto \delta_1(t,a) = t + a,$$

$$\lambda_1 : \mathbb{Z} \times \mathbb{Z} \longrightarrow FC_B$$

such that:

$$(t,a) \mapsto \lambda_1(t,a) : [1,+\infty[\rightarrow [1,+\infty[$$

defined, at a time, by:

$$\forall C \in B, \lambda_1(t,a)(C) = C^{(k^a)},$$

being k a real number > 1. Obviously,

$$[\lambda_1(0,1)]^a(C) = C^{(k^a)}.$$

6.10 Definition: $\overline{a}$-equivalence on Z

Let $\mathcal{L} = (Z \times B, A, \delta)$ be a financial law and $\overline{a} \in \overline{A}$. $\forall t, t' \in Z$, t and t' are called $\overline{a}$ - *equivalents* and it denotes

$$t \sim_{\overline{a}} t'$$

if

$$\overline{\lambda}_1(t,\overline{a}) \equiv_2 \overline{\lambda}_1(t',\overline{a}).$$

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6.11 Definition ($\overline{a}$-stationary financial law)

Let $\mathcal{L} = (Z \times B, A, \delta)$ be a financial law and $\overline{a} \in \overline{A}$. $\mathcal{L}$ is called $\overline{a}$ - *stationary* if $\forall t, t' \in Z, \overline{\lambda}_1(t, \overline{a}) \equiv_2 \overline{\lambda}_1(t', \overline{a})$, e. g., $\forall t, t' \in Z, t \sim_{\overline{a}} t'$.

Let us consider the following subset of $\overline{A}$, that we will denote as G:

$$G = \{\overline{a} \in \overline{A} / \mathcal{L} \textit{ is } \overline{a} - \textit{stationary}\}.$$

6.12 Theorem

$(G, \top)$ is an abelian group, being $\top$ the concatenation in A.

Proof.

1. $G \neq \emptyset$, because Λ (empty word) $\in G$:

$$\overline{\lambda}_1(t, \Lambda) = Id_B = \overline{\lambda}_1(t', \Lambda), \ \forall t, t' \in Z.$$

2. If $\overline{a}, \overline{a}' \in G$, then $\overline{aa'} = \overline{a} \top \overline{a}' \in G$.

3. In $(G, \top)$ it verifies the associative property, because it verifies in $(\overline{A}, \top)$.

4. In $(G, \top)$, the identity element is Λ.

5. $\forall \overline{a} = a_1 a_2 .....a_n \in G$, let us consider $-\overline{a} = (-a_1)(-a_2).....(-a_n)$.

$$\forall t, t' \in Z, \ \overline{\lambda}_1(t, -\overline{a}) =$$

$$= [\lambda_1(t - a_1, a_1)]^{-1} [\lambda_1(t - a_1 - a_2, a_2)]^{-1} .....[\lambda_1(t - a_1 - ..... - a_n, a_n)]^{-1} \equiv_2$$

$$\equiv_2 \lambda_1(t', -a_1)\lambda_1(t' - a_1, -a_2).....\lambda_1(t' - a_1 - ..... - a_{n-1}, a_n) =$$

$$= \overline{\lambda}_1(t', (-a_1)(-a_2).....(-a_n)) = \overline{\lambda}_1(t', -\overline{a}).$$

Therefore, $-\overline{a} \in G$.

The following theorem relates stationary financial laws and a-stationary financial laws.

6.13 Theorem

$\mathcal{L}$ is stationary if and only $\mathcal{L}$ is x-stationary, for all $x \in A$.

6.14 Example

Let $\mathcal{L} = (Z \times B, A, \delta)$ a financial law such that:

$$\forall t \in Z,\ \forall a \in A,\ \forall C \in B,\ \lambda_1(t,a)(C) = C.e^{\sum_{s=t}^{t+a-1} f(s)},$$

and

$$\lambda_1(t,0)(C) = C,$$

being $f(t)$ the function which verifies the following condition:

$$f(t) + f(t+1) + \ldots.. + f(t+n) = k,$$

where $f(t) > 0$ and $k > 0$ are constant.

The last equation is a equation in finite differences. For its resolution, we obtain the solutions of:

$$t^n + t^{n-1} + \ldots.. + t + 1 = 0.(*)$$

As $t^{n+1} - 1 = (t-1)(t^n + t^{n-1} + \ldots.. + t + 1)$ then the solutions of the equation:

$$t^n + t^{n-1} + \ldots.. + t + 1 = 0$$

are the n-th roots of the unity, except 1.

$$\sqrt[n+1]{1} = \sqrt[n+1]{1_{0^\circ}} = 1_{\frac{k.360^\circ}{n+1}},\ k = 0,\ 1,\ \ldots..,\ n.$$

Thus the solutions of the previous equation (*) are:

$$r_k = 1_{\frac{k.360^\circ}{n+1}},\ k = 1,\ \ldots..,\ n.$$

We consider two cases:

A) If n is even, we consider r_k, $k = 1, \ldots.., \frac{n}{2}$.

The solutions of the equation:

$$f(t) + f(t+1) + \ldots.. + f(t+n) = k\ (**)$$

are:

$$\left.\begin{cases} a_{kt}^{(1)} = \rho^t.cos(t\frac{k.360^\circ}{n+1}), \\ a_{kt}^{(2)} = \rho^t.sin(t\frac{k.360^\circ}{n+1}), \end{cases}\right\} k = 1,\ 2,\ \ldots..,\ \frac{n}{2},$$

e. g.:

$$\left.\begin{cases} a_{kt}^{(1)} = cos(t\frac{k.360^\circ}{n+1}), \\ a_{kt}^{(2)} = sin(t\frac{k.360^\circ}{n+1}), \end{cases}\right\} k = 1,\ 2,\ \ldots..,\ \frac{n}{2}.$$

Therefore the general solution of the equation (**) is:

$$f(t) = C_1.a_{1t}^{(1)} + C_2.a_{1t}^{(2)} + C_3.a_{2t}^{(1)} + C_4.a_{2t}^{(2)} +$$

$$+ \ldots\ldots + C_{n-1}.a_{\frac{n}{2}t}^{(1)} + C_n.a_{\frac{n}{2}t}^{(2)} + C_{n+1} \Leftrightarrow$$

$$\Leftrightarrow f(t) = C_1.cos(t\frac{1.360^o}{n+1}) + C_2.sen(t\frac{1.360^o}{n+1}) +$$

$$+ C_3.cos(t\frac{2.360^o}{n+1}) + C_4.sen(t\frac{2.360^o}{n+1}) +$$

$$+ \ldots\ldots + C_{n-1}.cos(t\frac{\frac{n}{2}.360^o}{n+1}) + C_n.sen(t\frac{\frac{n}{2}.360^o}{n+1}) + C_{n+1},$$

being C_1, C_2, ..., C_n arbitrary constants and $C_{n+1} = \frac{k}{n+1}$.

It verifies that $\mathcal{L}$ is a $(n+1)$-stationary financial law in ample sense. However, $\mathcal{L}$ is not an stationary financial law:

1.

$$\forall t \in Z;\ \forall a,\ a' \in A;\ \forall C \in B,$$

$$[\lambda_1(t+a; a') \circ \lambda_1(t,a)](C) = \lambda_1(t+a,a')[\lambda_1(t,a)(C)] =$$

$$= \lambda_1(t+a,a')[C.e^{\sum_{s=t}^{t+a-1} f(s)}] = C.e^{\sum_{s=t}^{t+a-1} f(s)}.e^{\sum_{s=t+a}^{t+a+a'-1} f(s)} =$$

$$= C.e^{\sum_{s=t}^{t+a+a'-1} f(s)} = \lambda_1(t, a+a')(C).$$

2.

$$\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+;\ x \neq 0;\ \forall C \in B,$$

$$\lambda_1(t,a+x)(C) = C.e^{\sum_{s=t}^{t+a+x-1} f(s)} >$$

$$> \text{(because of the definition, } f(s) > 0) > C.e^{\sum_{s=t}^{t+a-1} f(s)} = \lambda_1(t,a)(C).$$

3.

$$\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+;\ x \neq 0;\ \forall C \in B,$$

$$[\lambda_1(t,a+x) - \lambda_1(t,a)](C) = C.e^{\sum_{s=t}^{t+a+x-1} f(s)} - C.e^{\sum_{s=t}^{t+a-1} f(s)} =$$

$$= C.e^{\sum_{s=t}^{t+a-1} f(s)}[e^{\sum_{s=t+a}^{t+a+x-1} f(s)} - 1].$$

As the powers of e are bigger than 1, the product of two factors of C are bigger than 0, from which:

$$\lambda_1(t,a+x) - \lambda_1(t,a) \in FC_B.$$

Moreover, $\forall t \in Z$,

$$\lambda_1(t,n+1)(C) = C.e^{\sum_{s=t}^{t+n+1-1} f(s)} = C.e^{f(t)+f(t+1)+\ldots\ldots+f(t+n)} = C.e^k.$$

To depend $f(t)$ of n arbitrary constants, we can fix $f(t)$ choosing $k_1,\ k_2,\ .....,\ k_n$ such that:

$$k_1 \ + \ k_2 \ + \ ..... \ + \ k_{n+1} \ = \ k$$

and calculating the values of $C_1,\ C_2,\ .....,\ C_n$ that verify the following system of equations:

$$\left\{ \begin{array}{rcl} f(0) & = & k_1, \\ f(1) & = & k_2, \\ \vdots & & \vdots \\ f(n-1) & = & k_n, \end{array} \right.$$

and it is possible, because the following determinant:

$$\begin{vmatrix} cos(0\frac{1.360^o}{n+1}) & sin(0\frac{1.360^o}{n+1}) & \cdots & sin(0\frac{\frac{n}{2}.360^o}{n+1}) \\ cos(1\frac{1.360^o}{n+1}) & sin(1\frac{1.360^o}{n+1}) & \cdots & sin(1\frac{\frac{n}{2}.360^o}{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ cos((n-1)\frac{1.360^o}{n+1}) & sin((n-1)\frac{1.360^o}{n+1}) & \cdots & sin((n-1)\frac{\frac{n}{2}.360^o}{n+1}) \end{vmatrix}$$

is different of zero.

B) If n is odd, the general solution of the equation is:

$$\begin{aligned} f(t) \ &= \ C_1.a_{1t}^{(1)} \ + \ C_2.a_{1t}^{(2)} \ + \ C_3.a_{2t}^{(1)} \ + \ C_4.a_{2t}^{(2)} \ + \\ &+ \ ..... \ + \ C_{n-2}.a^{(1)}_{\frac{n-1}{2}t} \ + \ C_{n-1}.a^{(2)}_{\frac{n-1}{2}t} \ + \ C_n.(-1)^t \ + \ C_{n+1} \Leftrightarrow \\ \Leftrightarrow f(t) \ &= \ C_1.cos(t\frac{1.360^o}{n+1}) \ + \ C_2.sin(t\frac{1.360^o}{n+1}) \ + \\ &+ \ C_3.cos(t\frac{2.360^o}{n+1}) \ + \ C_4.sin(t\frac{2.360^o}{n+1}) \ + \\ &+ \ ..... \ + \ C_{n-2}.cos(t\frac{\frac{n-1}{2}.360^o}{n+1}) \ + \ C_{n-1}.sin(t\frac{\frac{n-1}{2}.360^o}{n+1}) \ + \\ &+ \ C_n.(-1)^t \ + \ C_{n+1}, \end{aligned}$$

being $C_1,\ C_2,\ .....,\ C_n$ arbitrary constants and $C_{n+1} = \frac{k}{n+1}$.

The discussion of this case is analogous to the previous.

6.15 Financial applications

1. Let us consider a financial entity that offer a t % as percentage of interest payable monthly, but variable.

   Thus this entity can establish the interests of the first months more low and the interests of the last months more high, for secure a period of time more elevated in the imposition.

2. Also this financial law can be applicated to the financing of vehicles or another movable goods, establishing a percentage of interest very low or zero in the first months and an interest very high in the last.

7 Conclusions

According to Levi (1973), a financial law is homogeneous of first-degree with regard to the quantity within certain limits, which induces to think in a homogeneity by "intervals of quantity". However, this problem has not been considered by this author and not another one, subsisting the generalized opinion of a difficult treatment of the question from the mathematical point of view.

In this work, we consider the problem from the point of view of the Algebraic Automata Theory, using an apparently sophisticated mathematical instrument though really intuitive and perfectly adequate to the problem in study. Note as the kernel of our financial law concept is the output function λ in which it remains fixed the temporal period of actuation (and not the expiry of the financial capitals). This implies an algebraic development of financial law concept, which permits to study the Financial Mathematics from a new point of view, generalizing classical concepts as the stationary financial systems.

We have defined a financial law as a "device" which transform capitals into capitals as consequence of the action of a temporal input on the first, ones producing a "change of state".

The financial law keeps satisfying some axioms and properties:

1. Strict increase according to the final moment.
2. Strict decrease according to the initial moment.
3. Strict increase according to the quantity.

But another axioms and properties are softened and it remain as follows:

1. The homogeneity of first-degree in accordance with the quantity isn't necessary.

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2. The continuity and, therefore, the derivability in accordance with the quantity aren't necessary.
3. The existence of an homeomorphism of the set of expirations into the set of quantities isn't necessary.

Moreover, we introduce another characteristic of the financial laws: its convexity or concavity, strict or no.

From the algebraic definition of financial law, we deduce some properties as:

1. The monoid of a financial law is a group, which, moreover, is isomorphic to the group of possible temporal inputs.
2. If a financial law isn't homogeneous of first-degree in accordance with the quantity, if we want operate with another money or monetary unit, we can find another law equivalent to the first which offer the same results when we cancel the change of money.

Successively we study the concepts of stationary and dynamic financial law, introducing an intermediate concept that is the stationary of order n financial law. This law has its origin in the changing but periodical character of some financial decisions on rates of interest, profitability, etc.. We justifie these laws as an incentive to place the capitals at certain moments of time.

Finally, we introduce the concept of $\overline{a}$-stationary financial law which also has its origin in a changing idea of the profitabilities, rates of interest, etc., but it includes the change of these not only in the origin but on the whole operation life time. This is refered to incentivate operations to place the capitals for certain fixed time period.

Notes

(1) A *financial system* is a function $F(C,t;p)$, where C is the quantity, t is the initial instant and p the final instant, which verifies the following conditions: homogeneity of degree one regard to the quantity C, $F(C,p;p)=C$, decrease according to t and increase according to p. If p is fixed, $F(C;t,p)$ is called a *financial law*.

(2) A system $F(C;t,p)$ is called *stationary* if $F(C;t,p)=F(C;t+h,p+h), \forall h \in \mathcal{R}$. In another case, $F(C;t,p)$ is called *dynamic*.

(3) A current account is called "highly remunerated" or "super-account" if it applies the compound interest with increasing rates of interest according to bands of quantities.

(4) A system $F(C;t,p)$ is called *simply multiplicative* (resp. *additive*) if $F(1;t,p).F(1;p,p')=F(1;t,p')$ (resp. $I(1;t,p)+I(1;p,p')=I(1;t,p')$, being $I(1;t,p)=F(1;t,p)-1$). Moreover, simply multiplicative (resp. additive) + stationary = *amply multiplicative* (resp. *additive*).

(5) In our study of automata, a *partial function* is a correspondence $f: A \longrightarrow B$ such that $f(a)=b$ and $f(a)=c$ implies $b=c$, e. g., doesn't need that $Dom(f)=A$.

(6) In fact, the complete property implies the reflexive property, from what this can be supressed in Axiom 1.

(7)

$$\tau_p(t_1,t_2;p)=\frac{F(t_1;p)-F(t_2;p)}{t_2-t_1},$$

being

$$F(t;p)=F(1,t;p).$$

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Table 1

YEARS	QUANTITIES				
	1.000	10.000	100.000	1.000.000	2.000.000
1	1.071	10.964	112.201	1.148.153	2.312.279
2	1.148	12.033	126.037	1.320.079	2.677.199
3	1.232	13.219	141.744	1.519.868	3.104.255
4	1.323	14.535	159.595	1.752.362	3.604.766
5	1.422	15.997	179.907	2.023.300	4.192.237
6	1.529	17.623	203.048	2.339.488	4.882.815
7	1.645	19.433	229.444	2.709.019	5.695.829
8	1.772	21.449	259.587	3.141.522	6.654.456
9	1.910	23.699	294.053	3.648.477	7.786.525
10	2.059	26.211	333.511	4.243.583	9.125.510
11	2.223	29.018	378.740	4.943.221	10.711.732
12	2.401	32.158	430.650	5.767.003	12.593.844
13	2.595	35.675	490.305	6.738.444	14.830.640
14	2.808	39.618	558.947	7.885.791	17.493.290
15	3.040	44.042	638.035	9.243.016	20.668.084
16	3.294	49.013	729.278	10.851.053	24.459.818

Table 2

YEARS	QUANTITIES				
	1.000	10.000	100.000	1.000.000	2.000.000
1	7'15	9'64	12'20	14'81	15'61
2	7'18	9'69	12'26	14'89	15'69
3	7'22	9'74	12'33	14'97	15'78
4	7'26	9'80	12'39	15'05	15'86
5	7'30	9'85	12'46	15'13	15'95
6	7'33	9'90	12'52	15'21	16'03
7	7'37	9'95	12'59	15'30	16'12
8	7'41	10'00	12'66	15'38	16'21
9	7'45	10'06	12'73	15'46	16'30
10	7'49	10'11	12'80	15'55	16'39
11	7'53	10'16	12'86	15'63	16'48
12	7'57	10'22	12'93	15'72	16'57
13	7'61	10'27	13'00	15'80	16'66
14	7'65	10'33	13'07	15'89	16'75
15	7'69	10'38	13'15	15'98	16'84
16	7'73	10'44	13'22	16'06	16'94

Table 3

DAYS	QUANTITIES				
	1.000	10.000	100.000	1.000.000	2.000.000
30	1.009	10.120	101.500	1.018.000	2.037.806
60	1.018	10.241	103.024	1.036.347	2.076.376
90	1.027	10.364	104.573	1.055.050	2.115.728
120	1.036	10.489	106.148	1.074.114	2.155.876
150	1.045	10.616	107.749	1.093.548	2.196.840
180	1.055	10.744	109.375	1.113.360	2.238.636
210	1.064	10.874	111.029	1.133.556	2.281.282
240	1.074	11.006	112.709	1.154.145	2.324.796
270	1.084	11.139	114.417	1.175.135	2.369.198
300	1.094	11.274	116.154	1.196.535	2.414.506
330	1.104	11.411	117.919	1.218.352	2.460.740
360	1.114	11.550	119.713	1.240.596	2.507.920

Table 4

DAYS	QUANTITIES				
	1.000	10.000	100.000	1.000.000	2.000.000
30	10’80	14’40	18’00	21’60	22’68
60	10’85	14’49	18’14	21,80	22’91
90	10’91	14’59	18’29	22’02	23’14
120	10’96	14’69	18’44	22’23	23’38
150	11’02	14’78	18’59	22’45	23’62
180	11’08	14’88	18’75	22’67	23’86
210	11’14	14’98	18’90	22’89	24’10
240	11’19	15’09	19’06	23’12	24’35
270	11’25	15’19	19’22	23’35	24’61
300	11’31	15’29	19’38	23’58	24’87
330	11’37	15’40	19’54	23’82	25’13
360	11’43	15’50	19’71	24’05	25’39

Figure 1

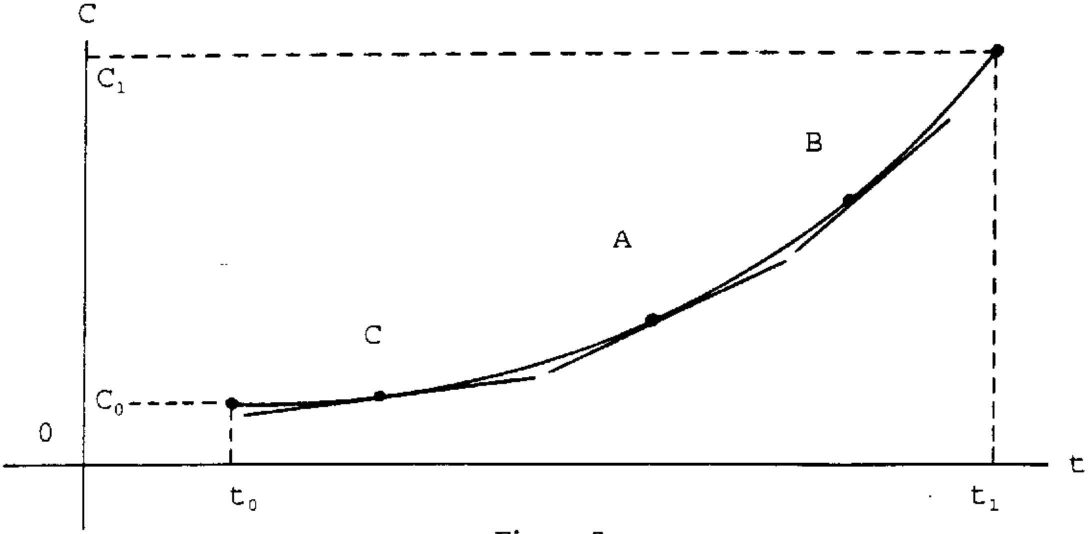

Figure 2

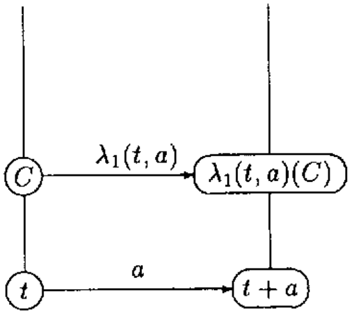

Figure 3